QUANTUM ERROR CORRECTION ON INFINITE-DIMENSIONAL HILBERT SPACES

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ABSTRACT. We present a generalization of quantum error correction to infinite-dimensional Hilbert spaces. The generalization yields new classes of quantum error correcting codes that have no finite-dimensional counterparts. The error correction theory we develop begins with a shift of focus from states to algebras of observables. Standard subspace codes and subsystem codes are seen as the special case of algebras of observables given by finite-dimensional von Neumann factors of type I. Our generalization allows for the correction of codes characterized by any von Neumann algebra and we give examples, in particular, of codes defined by infinite-dimensional algebras.

1. INTRODUCTION

A common challenge in the numerous fields of quantum information science is to devise techniques that protect the evolution of quantum systems from external perturbations. Since interactions with the environment are generally unavoidable, a variety of so-called quantum error correction procedures have been developed in order to correct the effect of environmental noise on quantum systems. The basic idea underlying the correction procedures is to exploit any knowledge that one may possess about the nature of the noise. To this end, the quantum information is usually encoded into a larger system in such a way that interactions with the environment affect essentially only the auxiliary degrees of freedom. In this way, the original qubits remain retrievable; i.e., the errors are correctable. So far, research in quantum error correction has been concerned primarily with the special case of a finite number of qubits which are embedded in a finite-dimensional Hilbert space. Real systems are, of course, always ultimately described in an infinite-dimensional Hilbert space. Algorithms for the cases of a finite or infinite number of qubits encoded in an infinite-dimensional Hilbert space have been proposed, in particular see [10, 26, 17, 12] respectively. However, no general theory (for instance, in the spirit of [20]) exists.

Here, we lay the foundations for quantum error correction in infinite-dimensional Hilbert spaces in full generality. We find that many of the basic results for quantum error correction extend to the infinite-dimensional setting, and that there are new phenomena which appear only in the infinite-dimensional setting. The new phenomena extend beyond obvious features

such as the possibility of an infinite number of individual errors. In particular, we uncover types of infinite-dimensional codes that have no finite-dimensional counterparts or approximations.

Historically, following the realization that quantum error correction is possible and the discovery of seminal examples [4, 30, 31, 16], Knill and Laflamme found a general mathematical condition characterizing the codes that are correctable for a given arbitrary noise model [20]. In this framework, the state of the information to be corrected is encoded in a subspace of the (finite-dimensional) physical Hilbert space. It was later realized [21, 23, 24] that there is a more subtle way of encoding the relevant information, namely in a *subsystem*. This amounts to assuming that we ignore the effect of the noise on the complementary subsystem. In finite-dimensions, this does not lead to larger quantum codes, but to more efficient correction procedures [29, 3, 1, 2]. It was shown in [6, 7] that this idea can be further generalized if, instead of a subsystem, one focuses on a restricted set of observables, which, for the purpose of quantum error correction, can always be assumed to span a finite-dimensional algebra. We will show that this idea naturally generalizes to infinite-dimensional Hilbert spaces, where the finite-dimensional algebras are replaced by arbitrary von Neumann algebras. In this context, a subsystem, as defined by a tensor-product structure on the underlying Hilbert space, corresponds to a von Neumann factor of type I. Other types of factors, however, also correspond to full-fledged logical quantum systems.

This paper is organized as follows. In the next section we give relevant background. We then begin our investigation, which includes an analysis of the sharp fixed and correctable observables; a derivation of the simultaneously correctable observables and operator systems; a brief review of von Neumann algebras and discussion of the special cases captured by the theory; presentation of explicit type I and type II examples; and finally an outlook section.

2. Background

2.1. **Quantum error correction.** We begin with a general review of quantum error correction motivated by the presentation of [21], which focused on the finite-dimensional case but equally well applies to the case of bounded interaction operators (as described below). Full details on our formalism in the infinite-dimensional setting will be provided below. Let us suppose that some information is sent through a quantum channel $\mathcal{E}$. The aim of quantum error correction is to find certain degrees of freedom, the *error correcting code* (whose exact nature we deliberately keep imprecise for the moment), on which the effect of the channel can be inverted. Since the inversion must be implemented physically, it must be a valid physical transformation; i.e., a channel. The inverse channel $\mathcal{R}$ is called the *correction channel*. Under reasonable assumptions (see Section 2.3) a channel $\mathcal{E}$ can

always be written as

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger,$$

which means that we can assume the noise is given by a discrete family of individual error operators E_i on Hilbert space.

In fact we will see that if $\mathcal{R}$ corrects this channel on some code, then it will correct also any channel whose elements span the same operator space as the elements of $\mathcal{E}$. This is important because often one does not know the precise channel elements E_i. Indeed, suppose that the system interacts continuously with its environment via a general Hamiltonian $H = H^\dagger$ of the form

$$(1) \qquad\qquad H = \sum_i J_i \otimes K_i$$

where the *interaction operators* J_i act on the system Hilbert space, and the operators K_i act on the environment.

In order for the state of the system at a given time to depend unambiguously on its initial state, we must assume that the initial state of the environment is uncorrelated with that of the system. For simplicity, we will further assume that it is a pure state $|\psi\rangle$. It is unlikely that we know much about K_i or $|\psi\rangle$ given that the environment may be very large and complex. However, it is generally conceivable that we have good knowledge of what the operators J_i can be. If $\mathcal{E}_t$ is the channel describing the evolution of the system alone up to time t, and $\mathcal{E}_t^*$ is the corresponding dual map, then we have

$$
\begin{aligned}
\mathcal{E}_t^*(A) &= (\mathbf{1} \otimes \langle\psi|)e^{itH}(A \otimes \mathbf{1})e^{-itH}(\mathbf{1} \otimes |\psi\rangle) \\
&= \sum_k (\mathbf{1} \otimes \langle\psi|)e^{itH}(\mathbf{1} \otimes |k\rangle)A(\mathbf{1} \otimes \langle k|)e^{-itH}(\mathbf{1} \otimes |\psi\rangle) \\
&= \sum_k E_k^\dagger(t)AE_k(t)
\end{aligned}
$$

(2)

where the channel elements are:

$$
\begin{aligned}
E_k(t) &= (\mathbf{1} \otimes \langle k|)e^{-itH}(\mathbf{1} \otimes |\psi\rangle) \\
&= \sum_n \frac{(-it)^n}{n!}(\mathbf{1} \otimes \langle k|)H^n(\mathbf{1} \otimes |\psi\rangle) \\
&= \sum_n \sum_{j_1 \ldots j_n} \frac{(-it)^n}{n!}\langle k|K_{j_1}\cdots K_{j_n}|\psi\rangle J_{j_1}\cdots J_{j_n}.
\end{aligned}
$$

(3)

Hence, we know that no matter what the environment operators and initial state are, the span of the channel elements $E_k(t)$ belongs to the algebra generated by the interaction operators J_i. One can look for correctable codes for all channels with this property. Such codes would be a form of infinite-distance code (in fact an example of noiseless subsystems [21]). While there

are important instances in which these codes do exist, they form a rather restrictive class of quantum error correcting code.

On the other hand, if the time t at which we aim to perform the correction is small enough, we can do better. Indeed, suppose that λ is some interaction parameter with unit of energy, then the above series is expressed in powers of $t\lambda$. We see that to the nth order in $t\lambda$, the elements of the channel are in the span of the nth order products $J_{j_1} \cdots J_{j_n}$. Hence, if we correct often enough (in order to limit the value of $t\lambda$), then we only need to find a correction channel and a code for channel elements in the span of the operators $J_{j_1} \cdots J_{j_n}$, $n < N$, for a fixed N. In this context, the operators J_i are seen as representing individual errors, and our code corrects up to N independent errors [21].

For clarity of the presentation we will stick to the simple picture where the channel $\mathcal{E}$ is given. However we will keep the more realistic situation in mind and check that the correction procedure we devised works not just for the given channel, but also for any channel whose elements span the same space.

2.2. **Stochastic Heisenberg picture.** Traditionally, one attempts to correct *states*, namely to simultaneously find a state ρ and a channel $\mathcal{R}$ such that $\mathcal{R}(\mathcal{E}(\rho)) = \rho$. In [6, 7], it was shown that it is convenient to consider instead the correction of *observables*. Most generally, an observable is specified by a positive operator valued measure (POVM). For instance, if the measure is discrete, the POVM X is specified by a family of positive operators X_k called *effects*. In general an effect can be any positive operator smaller than the identity: $0 \leq X_k \leq \mathbf{1}$. In order to form a discrete POVM, these effects must sum to the identity: $\sum_k X_k = \mathbf{1}$. More generally, a POVM X sends measurable subsets of a measure space Ω to positive operators. It is defined by the effects $X(\omega)$, $\omega \subseteq \Omega$. For a discrete POVM, $X(\omega) = \sum_{k \in \omega} X_k$.

What matters physically are the expectation values of the form

$$\mathrm{Tr}(\rho A)$$

where A is an effect. Indeed, for a POVM X, $\mathrm{Tr}(\rho X(\omega))$ is the probability that the outcome of a measurement of X falls inside $\omega \subseteq \Omega$.

If the channel $\mathcal{E}$ describes the evolution of our system, a measurement of an observable X after the evolution will yield probabilities of the form $\mathrm{Tr}(\mathcal{E}(\rho)X(\omega))$. Alternatively, one can define a dual map $\mathcal{E}^*$ which satisfies

$$\mathrm{Tr}(\mathcal{E}(\rho)X(\omega)) = \mathrm{Tr}(\rho\,\mathcal{E}^*(X(\omega))$$

for all $\omega \subseteq \Omega$. This dual channel is the stochastic form of the usual Heisenberg picture of quantum mechanics. It specifies the evolution of observables instead of states. In this picture, an observable X evolves to the observable Y defined by $Y(\omega) = \mathcal{E}^*(X(\omega))$. An important conceptual point to note here, which is not apparent in the usual case of unitary evolutions, is that this evolution goes "backward" in time. Indeed, if the channel $\mathcal{E}_2$ follows the channel $\mathcal{E}_1$ in time, the overall transformation in the Schrödinger picture is

given by the channel $\mathcal{E}_2 \circ \mathcal{E}_1$. But the dual channels compose in the reverse order; we have $(\mathcal{E}_2 \circ \mathcal{E}_1)^* = \mathcal{E}_1^* \circ \mathcal{E}_2^*$. This means that the latest transformation must be applied first.

Instead of correcting states, we may thus attempt to correct observables; i.e., to pair a channel $\mathcal{R}$ with an observable X such that

$$(\mathcal{R} \circ \mathcal{E})^*(X(\omega)) = X(\omega)$$

for all $\omega \subseteq \Omega$. This expression means that measuring X before or after the action of the map $\mathcal{R} \circ \mathcal{E}$ would yield the same outcomes with the same probabilities no matter what the initial state was. Hence we can say that the observable X is corrected by $\mathcal{R}$ for the noise defined by $\mathcal{E}$. In general, if the channel $\mathcal{R}$ exists such that this equation is satisfied, we say that the observable X is *correctable*. The sets of simultaneously correctable *sharp* observables, that is correctable by a single common channel $\mathcal{R}$, were characterized in [6] and shown to generalize all previous known types of quantum error correcting codes. We will show here that this approach leads naturally to an infinite-dimensional generalization.

2.3. **States and channels in infinite dimensions.** We consider a quantum system characterized by a Hilbert space $\mathcal{H}$ that may be infinite-dimensional. For effects, it is natural to consider the *bounded* linear operators on $\mathcal{H}$. Indeed, the condition $0 \leq A \leq \mathbf{1}$ guarantees that A is a bounded operator on $\mathcal{H}$. We denote the set of bounded operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$, and the set of effects (or positive contractions on $\mathcal{H}$) by $\mathbb{E}(\mathcal{H})$. The set $\mathcal{B}(\mathcal{H})$ is naturally a von Neumann algebra. In general we could assume that states are positive linear functionals on $\mathcal{B}(\mathcal{H})$. In the finite dimensional case, this guarantees that they are of the form $A \mapsto \text{Tr}(\rho A)$ for some operator ρ and all effects A. However, this does not carry through to the infinite-dimensional case. Instead, we postulate that a state is represented by a positive operator ρ such that $\text{Tr}(\rho A)$ is well defined for any effect A, and also such that $\text{Tr}(\rho) = 1$. This means that states are *trace-class* operators. Formally, the trace-class operators $B \in \mathcal{B}(\mathcal{H})$ are those for which the expression

$$\sum_i \langle i | \sqrt{B^\dagger B} | i \rangle$$

converges for (one and hence) any basis $|i\rangle$. We will let $\mathcal{B}_t(\mathcal{H})$ denote the set of trace-class operators on $\mathcal{H}$. For self-adjoint elements $\rho \in \mathcal{B}_t(\mathcal{H})$, we have a *trace* defined by

$$\text{Tr}(\rho) := \sum_i \langle i | \rho | i \rangle.$$

The product of an element of $\rho \in \mathcal{B}_t(\mathcal{H})$ with any operator $A \in \mathcal{B}(\mathcal{H})$ is also trace-class, which implies that we can define $\text{Tr}(\rho A)$. The set $\mathcal{B}_t(\mathcal{H})$ itself is a Banach algebra. It is the pre-dual of $\mathcal{B}(\mathcal{H})$, which means that $\mathcal{B}(\mathcal{H})$ is the set of linear functionals on $B_t(\mathcal{H})$. This means that effectively we have defined the effects as linear functionals on states rather than the

converse. The existence of a pre-dual is a fundamental property of von Neumann algebras.

A general von Neumann algebra is equipped with a weak-$*$ topology, which amounts to defining the convergence of a sequence $A_n \in \mathcal{B}(\mathcal{H})$ in terms of expectation values. In the case of $\mathcal{B}(\mathcal{H})$, this means that the sequence converges to A if and only if the numbers $\mathrm{Tr}(\rho A_n)$ converge to $\mathrm{Tr}(\rho A)$ as $n \to \infty$ for all states $\rho \in \mathcal{B}_t(\mathcal{H})$.

This implies that the states represented by elements of $\mathcal{B}_t(\mathcal{H})$, seen as linear functionals of effects, are continuous with respect to the weak-$*$ topology. Indeed, if the sequence $\{A_n\}_{n=1}^\infty$ converges to A in this topology, then, by definition $\mathrm{Tr}(\rho A_n) \to \mathrm{Tr}(\rho A)$. Hence the map $A \mapsto \mathrm{Tr}(\rho A)$ is continuous. Conversely, those are *all* the weak-$*$ continuous positive linear functionals. Therefore, our choice of states corresponds to restricting the natural set of all linear functionals on effects to only those which are weak-$*$ continuous, or *normal* for short.

A channel from a system represented by the Hilbert space $\mathcal{H}_1$ to a system represented by $\mathcal{H}_2$ can be defined by a trace-preserving completely positive linear map

$$\mathcal{E} : \mathcal{B}_t(\mathcal{H}_1) \to \mathcal{B}_t(\mathcal{H}_2)$$

on states. It has a dual

$$\mathcal{E}^* : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$$

which describes the evolution of effects, and hence observables, in the Heisenberg picture. A channel can be represented as

$$\mathcal{E}^*(A) = \sum_{k=1}^\infty E_k^\dagger A E_k$$

or

$$\mathcal{E}(\rho) = \sum_{k=1}^\infty E_k \rho E_k^\dagger$$

where the sum can now be infinite [22]. We will call this the operator-sum form of the channel $\mathcal{E}$. The elements E_k are bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. This can be understood starting from the Stinespring dilation theorem for completely positive maps between C*-algebras, which states that there is a representation π of $\mathcal{B}(\mathcal{H}_2)$ on some Hilbert space $\mathcal{K}$, and an isometry $V : \mathcal{H}_1 \to \mathcal{K}$, such that

$$\mathcal{E}^*(A) = V^\dagger \pi(A) V$$

for all $A \in \mathcal{B}(\mathcal{H}_2)$. In the case that we are considering (von Neumann factor of type I, and normal map), the representation π on $\mathcal{H}$ is of the form $\pi(A) = A \otimes \mathbf{1}$. Also, if $\mathcal{H}_2$ is separable, then so is $\mathcal{K}$ [28]. Therefore the subsystem on which $\pi(A)$ acts trivially is also separable, and possesses a

discrete basis $|i\rangle$. This implies that

$$\mathcal{E}^*(A) = V^\dagger(A \otimes \mathbf{1})V = \sum_i V^\dagger(A \otimes |i\rangle\langle i|)V$$

$$= \sum_i (V^\dagger|i\rangle)A(\langle i|V),$$

where the operators $\langle i|V : \mathcal{H}_1 \to \mathcal{H}_2$ are defined by the induced tensor structure on the dilation space $\mathcal{K}$. Thus the elements of the channel $\mathcal{E}$ can be chosen to be $E_i = \langle i|V$. Note from this observation it is clear there is a large ambiguity in the choice of the elements E_i. Indeed, any orthonormal basis $|i\rangle$ would potentially yield a different set of elements.

3. Sharp Correctable Observables

A POVM X is correctable if the effect of the channel can be inverted on all the effects $X(\omega)$ in the following sense.

Definition 3.1. We say that an effect $0 \leq A \leq \mathbf{1}$ is *correctable* for the channel $\mathcal{E}$ if there exists a channel $\mathcal{R}$ such that

$$A = \mathcal{E}^*(\mathcal{R}^*(A))$$

In this section we will characterize the correctability of certain types of effects, namely projectors. This is important because the effects of a *sharp observable*; i.e., a traditional observable represented by a self-adjoint operator, are always projectors (they are the spectral projectors of the corresponding self-adjoint operator). The results obtained in this section will form the basis of our understanding of correctable observables. We begin with a "warm up", the case of passive error correction in this setting.

3.1. **Sharp fixed observables.** We consider the problem of characterizing the sharp observables that are unaffected by the action of the channel; that is, which are correctable in the above sense but with the trivial correction channel $\mathcal{R}(\rho) = \rho$. This requires that we use the same source and destination Hilbert spaces, namely

$$\mathcal{H} := \mathcal{H}_1 = \mathcal{H}_2.$$

Definition 3.2. We say that an observable X is *fixed* by the channel $\mathcal{E}$ if

$$X(\omega) = \mathcal{E}^*(X(\omega)) \quad \text{for all } \omega.$$

A slightly more general form of this problem (see below) was addressed for channels defined on finite-dimensional Hilbert spaces in [6] and shown to yield all noiseless subsystems [21, 19, 13]. These results may be readily generalized to the infinite-dimensional setting, provided that we model the proof on the approach of [7] which does not refer to the structure theory of finite-dimensional algebras.

Since we focus on sharp observables, the effects $X(\omega)$ are all projectors. Let us therefore first characterize the fixed projectors P, which satisfy $P =$

$\mathcal{E}^*(P)$. By multiplying on both sides by the orthogonal projector $P^\perp = \mathbf{1} - P$, we obtain $0 = P^\perp \mathcal{E}^*(P) P^\perp = \sum_k P^\perp E_k^\dagger P E_k P^\perp$. The right hand side is a sum of positive operators, which must therefore all equal zero: $(P E_k P^\perp)^\dagger P E_k P^\perp = 0$ for all k, which in turns implies

$$(4) \qquad\qquad\qquad P E_k P^\perp = 0$$

for all k. Similarly, because the dual channel is always unital, we have $P^\perp = \mathcal{E}^*(P^\perp)$, from which we also deduce

$$(5) \qquad\qquad\qquad P^\perp E_k P = 0.$$

Combining Eq. (4) and Eq. (5), we obtain $P E_k = E_k P$ for all k. Hence the fixed projectors must commute with all the channel elements. This condition is in fact sufficient since it implies

$$\mathcal{E}^*(P) = \mathcal{E}^*(\mathbf{1}) P = P.$$

Hence we have proved the following.

Proposition 3.1. *A projector P satisfies $\mathcal{E}^*(P) = P$ if and only if it commutes with every channel element for $\mathcal{E}^*$;*

$$(6) \qquad\qquad\qquad [P, E_k] = 0 \quad \text{for all } k.$$

We see that the commutant algebra of the channel elements, given by

$$\mathcal{A}^N := \{A \in \mathcal{B}(\mathcal{H}) : [A, E_k] = [A^\dagger, E_k] = 0 \,\forall k\},$$

which is a von Neumann algebra, plays a central role here. Indeed, since a von Neumann algebra is spanned by its projectors, $\mathcal{A}^N$ is spanned by all the fixed projectors. In addition, the channel clearly fixes all the elements of $\mathcal{A}^N$. We will call $\mathcal{A}^N$ the *noiseless algebra* for $\mathcal{E}$.

Since a sharp observable X is fixed by definition when all of its elements $X(\omega)$ are fixed, we have from Proposition 3.1 that it is fixed if and only if its spectral projectors $X(\omega)$ belong to $\mathcal{A}^N$, which is equivalent to simply asking that the corresponding self-adjoint operator $\widehat{X}$ belongs to $\mathcal{A}^N$.

Proposition 3.2. *A sharp observable X represented by the self-adjoint operator $\widehat{X}$ is fixed by $\mathcal{E}$ if and only if $\widehat{X} \in \mathcal{A}^N$; i.e.,*

$$[\widehat{X}, E_k] = 0 \quad \text{for all } k.$$

We will show in Section 5.2, in the more general setting were the correction procedure may be non-trivial, that when the algebra $\mathcal{A}^N$ is a factor of type I, it corresponds to a noiseless subsystem. All types of factors can emerge in this way. Type II and III factors define new types of noiseless subsystems which have no finite-dimensional counterpart. An example of a noiseless factor of type II will be given in Section 7.

Let us remark that we can obtain more noiseless algebras by considering the sharp observables which are fixed only provided a certain restriction on states characterized by a subspace $\mathcal{H}_0 \subseteq \mathcal{H}$. Let us introduce the isometry

$$V : \mathcal{H}_0 \to \mathcal{H}$$

which embeds $\mathcal{H}_0$ into $\mathcal{H}$, that is, $V^\dagger V = \mathbf{1}_{\mathcal{H}_0}$ and $VV^\dagger$ is the projector of $\mathcal{H}$ onto $\mathcal{H}_0$. We write $\mathcal{E}_0$ for the channel $\mathcal{E}$ restricted to the states in the subspace $\mathcal{H}_0$,

$$\mathcal{E}_0(\rho) := \mathcal{E}(V\rho V^\dagger).$$

This channel has elements $E_k V$. We cannot directly apply Proposition 3.2 because this channel does not have the same source and destination spaces. In order to define what it means for an observable of $\mathcal{H}_0$ to be fixed by $\mathcal{E}_0$ we have to specify how we will map back the output of $\mathcal{E}_0$ from $\mathcal{H}$ to $\mathcal{H}_0$. The most natural way to do this is to apply the dual of the isometry. Other possibilities would correspond to different type of "corrections", and would enter the more general framework presented in the next section.

Hence we must characterize the projectors P satisfying

$$\mathcal{E}_0^*(VPV^\dagger) = P.$$

Note that the map $X \mapsto \mathcal{E}_0^*(V^\dagger X V)$ is not unital so that we cannot apply Proposition 3.2 directly to it either. However, a small variation of the steps followed in the proof of Proposition 3.2 yields the necessary and sufficient condition

$$(VPV^\dagger)E_k V = E_k V P.$$

We leave the details to the interested reader. The noiseless algebra here is

$$\mathcal{A}_0^N = \{A \in \mathcal{B}(\mathcal{H}_0) : VAV^\dagger E_k V = E_k V A,$$
$$V A^\dagger V^\dagger E_k V = E_k V A^\dagger \ \forall k\}.$$

Thus each subspace of the Hilbert space is associated with a different noiseless algebra.

If $\mathcal{H}$ is finite-dimensional, the noiseless algebras that are factors correspond precisely to the noiseless subsystems [6]. The problem in the characterization of noiseless subsystems for finite-dimensional Hilbert spaces lies with the identification of the subspaces which support a nontrivial noiseless algebra.

3.2. **Sharp correctable observables.** We move now to the case of arbitrary correction operations. If A is an effect such that Definition 3.1 holds, then there exists an effect B such that $A = \mathcal{E}^*(B)$. Let us consider the correctable sharp effects. A sharp effect $P \in \mathcal{B}(\mathcal{H}_1)$ is a an effect which is also a projection; $P^2 = P = P^\dagger$. If it is correctable, then there is an effect $B \in \mathcal{B}(\mathcal{H}_2)$ such that

(7) $$P = \mathcal{E}^*(B).$$

In order to characterize the correctable sharp effects, we want to obtain an equivalent condition which involves only the channel elements E_k. To do this, we multiply Eq. (7) from both sides by $P^\perp = \mathbf{1} - P$ to obtain

$$P^\perp \mathcal{E}^*(B)P^\perp = P^\perp P P^\perp = 0.$$

This implies $BE_k(\mathbf{1} - P) = 0$ for all k; i.e.,

$$(8) \qquad\qquad BE_k = BE_kP.$$

There is another similar equation that we can use. Indeed, $\mathcal{E}^*(\mathbf{1} - B) = \mathbf{1} - \mathcal{E}^*(B) = \mathbf{1} - P$, and $\mathbf{1} - B$ is a valid effect. Using the same trick as above, we obtain $(\mathbf{1} - B)E_kP = 0$; i.e.,

$$(9) \qquad\qquad E_kP = BE_kP.$$

Combining Eq. (8) and Eq. (9), we get

$$(10) \qquad\qquad BE_k = E_kP.$$

This means than the existence of an effect B which satisfies $BE_k = E_kP$ for all channel elements E_k is a necessary condition for P to be correctable. Note that this condition also implies $\mathcal{E}^*(B) = P$, since

$$\mathcal{E}^*(B) = \sum_k E_k^\dagger B E_k = \sum_k E_k^\dagger E_k P = \mathcal{E}^*(\mathbf{1})P = P.$$

Thus we have obtained a condition that depends on the channel elements E_k independently rather than on the whole channel. However, a much more useful characterization of the correctable effects requires the elimination of any reference to the unknown operator B. First note that by taking the adjoint of Eq. (10) we obtain $E_k^\dagger B = P E_k^\dagger$. Together with Eq. (10), this implies

$$E_k^\dagger E_j P = E_k^\dagger B E_j = P E_k^\dagger E_j.$$

This is a necessary condition for P to be correctable which does not involve the unknown effect B. Algebraically, this condition is precisely the one found in [6] for the finite-dimensional case. In the following result we show this condition is also sufficient in the infinite-dimensional case, which requires a more delicate analysis.

Theorem 3.3. *A sharp effect P is correctable for the channel $\mathcal{E}(\rho) = \sum_k E_k\rho E_k^\dagger$ if and only if*

$$(11) \qquad\qquad [P, E_i^\dagger E_j] = 0 \text{ for all } i,j.$$

Proof. We have already proven the necessity. In order to prove the sufficiency, we will use this condition to build the effect B of which P is the image. We will need the completely positive map $\mathcal{E}_\lambda$ defined by

$$(12) \qquad\qquad \mathcal{E}_\lambda(A) := \sum_{i=0}^{\infty} \lambda_i E_i A E_i^\dagger$$

where $\lambda_i := 2^{-i}$. This choice of λ guarantees that the sum converges in norm for any effect A since $\|E_i A E_i^\dagger\| \leq 1$. In fact, any choice of components $\lambda_i > 0$ which makes this sum weak-$*$ convergent for any effect A would be sufficient for our purpose. Note that if $\lambda_i = 1$ for all i, then $\mathcal{E}_\lambda = \mathcal{E}$, which is defined only on trace-class operators.

We will try the ansatz

$$(13) \qquad B := (\mathcal{E}_\lambda(\mathbf{1}))^{-1} \mathcal{E}_\lambda(P).$$

First, we have to show that $\mathcal{E}_\lambda(\mathbf{1})$ can indeed be inverted. Note that $\mathcal{E}_\lambda(\mathbf{1})|\psi\rangle = 0$ if and only if $\langle\psi|\mathcal{E}_\lambda(\mathbf{1})|\psi\rangle = 0$ since it is positive. If we write the channel in terms of the elements E_k, we obtain a sum of positive terms which must all be equal to zero: $\langle\psi|E_i E_i^\dagger|\psi\rangle = 0$. Hence $E_i^\dagger|\psi\rangle = 0$ for all i. This means that if $|\psi\rangle$ is in the kernel of $\mathcal{E}_\lambda(\mathbf{1})$, it must be in the kernel of each $E_i^\dagger$, and therefore be orthogonal to the range of each E_i. This shows that we can invert $\mathcal{E}_\lambda(\mathbf{1})$ on the range of any of the operators E_i. In the following we will always be able to assume that $(\mathcal{E}_\lambda(\mathbf{1}))^{-1}$ operates on the span of the ranges of the operators E_i. In particular, Eq. (13) is well-defined.

Now that we have defined B, let us check that $\mathcal{E}^*(B) = P$. We have

$$\begin{aligned}
BE_i &= (\mathcal{E}_\lambda(\mathbf{1}))^{-1} \mathcal{E}_\lambda(P) E_i \\
&= (\mathcal{E}_\lambda(\mathbf{1}))^{-1} \sum_k \lambda_k E_k P E_k^\dagger E_i \\
&= (\mathcal{E}_\lambda(\mathbf{1}))^{-1} \sum_k \lambda_k E_k E_k^\dagger E_i P \\
&= (\mathcal{E}_\lambda(\mathbf{1}))^{-1} \mathcal{E}_\lambda(\mathbf{1}) E_i P = E_i P,
\end{aligned}$$

which is Eq. (10) and proves that $\mathcal{E}^*(B) = P$. However we have to check that B is an effect (i.e. a positive contraction). Note that $BE_i = E_i P$ implies $\sum_i \lambda_i B E_i E_i^\dagger = \sum_i \lambda_i E_i P E_i^\dagger$. In other words $B\mathcal{E}_\lambda(\mathbf{1}) = \mathcal{E}_\lambda(P)$, which in turn yields $B = \mathcal{E}_\lambda(P)(\mathcal{E}_\lambda(\mathbf{1}))^{-1}$. From the definition of B, we also have $B = (\mathcal{E}_\lambda(\mathbf{1}))^{-1}\mathcal{E}_\lambda(P)$. Hence we have $[(\mathcal{E}_\lambda(\mathbf{1}))^{-1}, \mathcal{E}_\lambda(P)] = 0$, which implies that

$$(14) \qquad B = \mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}} \mathcal{E}_\lambda(P) \mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}} \geq 0$$

In addition, $B \leq \mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}} \mathbf{1} \mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}} = \mathbf{1}$, which shows that B is an effect. Moreover, B is obtained from P by the map $\mathcal{R}^*$ defined by

$$\mathcal{R}^*(A) = \mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}} \mathcal{E}_\lambda(A) \mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}}.$$

This map is positive by construction, and we have already shown that it is unital. It is the dual of the quantum channel

$$(15) \qquad \mathcal{R}(\rho) = \mathcal{E}_\lambda^*(\mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}} \rho \, \mathcal{E}_\lambda(\mathbf{1})^{-\frac{1}{2}}),$$

which therefore is the correction channel for all correctable sharp observables. $\square$

If $\mathcal{H}_1$ is finite-dimensional, one can use $\lambda_i = 1$. In this case, Eq. (15) reduces to the explicit way of writing the correction channel used in [6]. This explicit form has also appeared in [9].

Theorem 3.3 also yields some unsharp correctable effects. Indeed, given the linearity of $\mathcal{E}^*$, if two effects are corrected by the same channel $\mathcal{R}$, then

so is any of their convex combinations (which are also effects). Therefore, the convex hull of the correctable projectors is entirely correctable. In fact, the continuity of this channel implies that the weak-∗ closure of this convex hull is also correctable.

Consider the commutant $\mathcal{A}$ of the operators $E_i^\dagger E_j$; i.e., the set of operators that commute with each of them,

$$\mathcal{A} := \{A : [A, E_i^\dagger E_j] = 0 \ \forall \ i, \ j\}.$$

This set is clearly an algebra. In fact, it is a von Neumann algebra, which always has the property that the set of effects it contains is the closed convex hull of its projectors [14]. Since all the projectors in $\mathcal{A}$ are corrected by $\mathcal{R}$, then so are all the effects it contains. This proves the following.

Corollary 3.4. *The set of effects spanning the von Neumann algebra*

$$(16) \qquad \mathcal{A} = \{A \in \mathcal{B}(\mathcal{H}_1) \ : \ [A, E_i^\dagger E_j] = 0 \text{ for all } i, j\}$$

are all corrected by the channel defined in Eq. (15). In addition, this algebra contains all the correctable sharp effects.

We now formalize the notion of a correctable positive operator-valued measure.

Definition 3.3. We say that a POVM X is *correctable* for $\mathcal{E}$ if there is a channel $\mathcal{R}$, called the *correction channel*, which is such that

$$X(\omega) = \mathcal{E}^*(\mathcal{R}^*(X(\omega)))$$

for all $\omega \subseteq \Omega$. We then say that X is *fixed* by the channel $\mathcal{R} \circ \mathcal{E}$.

The operational meaning of this definition is clear. If X is fixed by $\mathcal{R} \circ \mathcal{E}$, then a measurement of X after the application of the channel $\mathcal{E}$ followed by $\mathcal{R}$ will yield the same outcome as if nothing had happened to the system.

Since the effects associated with a sharp observable are all projectors, Corollary 3.4 tells us that the correctable sharp observables are precisely those which are represented by a self-adjoint operator in the algebra $\mathcal{A}$.

Theorem 3.5. *A sharp observable X, represented by the self-adjoint operator $\widehat{X}$, is correctable for $\mathcal{E}$ if and only if $\widehat{X} \in \mathcal{A}$.*

4. Simultaneously Correctable Observables and Operator Systems

We know that, in fact, sharp correctable observables can all be corrected with the same correction channel. Let us explicitly define this concept.

Definition 4.1. A set of POVMs is said to be *simultaneously correctable* for $\mathcal{E}$ if every POVM in the set is correctable by the same correction channel $\mathcal{R}$, which means that they are all fixed by $\mathcal{R} \circ \mathcal{E}$.

Such a set is what constitutes our most general notion of a *quantum error correcting code.* These simultaneously correctable observables represent a consistent set of properties of the system which can be saved from the noise. However, instead of working with a set of POVMs, it is more convenient to work with their effects.

Clearly, if a set of effects is simultaneously correctable, then so are all linear combinations which are also effects. Since, in addition, effects are self-adjoint, this means that a set of simultaneously correctable effects will always be characterized by an *operator system* $\mathcal{S}$; i.e., a linear subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}_1)$ which contains the adjoint of all its elements [28]. Hence, if we say that an operator system $\mathcal{S}$ is correctable, we mean that all the effects in it are correctable. In fact, given that we are considering normal channels, it follows that any weak-$*$ limit of such linear combination will be correctable by the same correction channel. Hence we shall consider weak-$*$ closed operator systems.

Therefore our most general notion of a quantum code is a weak-$*$ closed operator system generated by a set of simultaneously correctable effects $\mathcal{S}$. Clearly, $\mathcal{S}$ represents a set of simultaneously correctable observables, namely all the observables X with effects $X(\omega) \in \mathcal{S}$ for all ω.

Our task is to characterize the maximal correctable operator systems for a channel $\mathcal{E}$. From the previous section we already know one: the algebra $\mathcal{A}$ defined in Eq. (16).

Proposition 4.1. *The POVMs with effects in $\mathcal{A}$ are all simultaneously correctable by the channel defined in Eq. (15).*

We will call $\mathcal{A}$ *the correctable algebra* for $\mathcal{E}$. This is justified by the fact that any other algebra spanned by correctable effects is inside $\mathcal{A}$. Indeed, such an algebra would be spanned by its projectors, but $\mathcal{A}$ already contains all the correctable projectors.

We will now show that many other correctable operator systems can be obtained via Proposition 4.1. Consider any subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$, and let

$$V : \mathcal{H}_0 \rightarrow \mathcal{H}_1$$

be the isometry which embeds $\mathcal{H}_0$ into $\mathcal{H}_1$. This means that the operator $P_0 := VV^\dagger$, defined on $\mathcal{H}_1$, is the projector on the subspace $\mathcal{H}_0$, whereas $V^\dagger V = \mathbf{1}_0$ is the identity inside $\mathcal{H}_0$. We can define a new channel $\mathcal{E}_0$ by restricting the channel $\mathcal{E}$ to the subspace $\mathcal{H}_0$, which, physically, amounts to making sure that the initial state is prepared inside $\mathcal{H}_0$. Hence we define

$$\mathcal{E}_0(\rho) := \mathcal{E}(V \rho V^\dagger)$$

whose dual is

$$\mathcal{E}_0^*(A) = V^\dagger \mathcal{E}^*(A) V$$

This channel has its own correctable algebra:

$$\mathcal{A}_0 = \{ A \in \mathcal{B}(\mathcal{H}_0) \ : \ [A, V^\dagger E_k^\dagger E_l V] = 0 \}.$$

This algebra can be naturally embedded in $\mathcal{B}(\mathcal{H}_1)$ via the isometry V; the map $A \mapsto VAV^\dagger$ from $\mathcal{B}(\mathcal{H}_0)$ to $\mathcal{B}(\mathcal{H}_1)$ is a normal $*$-homomorphism. Note however that the identity on $\mathcal{B}(\mathcal{H}_0)$ is sent to the projector $VV^\dagger = P$.

The algebra $V\mathcal{A}_0V^\dagger$ may or may not be a subalgebra of the correctable algebra $\mathcal{A}$. If it is not, then its effects are not correctable for the channel $\mathcal{E}$ itself. However, we will see that it is one-to-one with a family of simultaneously correctable effects which do not form an algebra. Indeed, let $\mathcal{R}_0$ be the correction channel for $\mathcal{E}_0$, which is a channel from $\mathcal{B}_t(\mathcal{H}_2)$ to $\mathcal{B}_t(\mathcal{H}_0)$, and define the set

$$\text{(17)} \qquad \mathcal{S}_0 := \mathcal{E}^*(\mathcal{R}_0^*(\mathcal{A}_0)).$$

Note that $\mathcal{S}_0$ not equal to $\mathcal{A}_0$ because we have used $\mathcal{E}^*$ instead of $\mathcal{E}_0^*$. In fact, we have

$$\text{(18)} \qquad \mathcal{A}_0 = V^\dagger \mathcal{S}_0 V.$$

This subset $\mathcal{S}_0 \subset \mathcal{B}(\mathcal{H}_1)$ is in general not an algebra, but it is always an operator system. We claim that all the effects which are in $\mathcal{S}_0$ are simultaneously correctable. Indeed, for any effect $A = \mathcal{E}^*(\mathcal{R}_0^*(B)) \in \mathcal{S}_0$, where $B \in \mathcal{A}_0$, we have

$$\begin{aligned} \mathcal{E}^*(\mathcal{R}_0^*(V^\dagger AV)) &= \mathcal{E}^*(\mathcal{R}_0^*(V^\dagger \mathcal{E}^*(\mathcal{R}_0^*(B))V)) \\ &= \mathcal{E}^*(\mathcal{R}_0^*(\mathcal{E}_0^*(\mathcal{R}_0^*(B)))) \\ &= \mathcal{E}^*(\mathcal{R}_0^*(B)) = A. \end{aligned}$$

Hence, the correction map is

$$\rho \mapsto V\mathcal{R}_0(\rho)V^\dagger,$$

which is a valid channel from $\mathcal{B}_t(\mathcal{H}_2)$ to $\mathcal{B}_t(\mathcal{H}_1)$.

This shows that, for any subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$ we can construct an operator system S_0 whose effects are all simultaneously correctable. In addition, it is clear that all the observables which are formed from effects in S_0 are simultaneously correctable. We have thus established the following result.

Theorem 4.2. *For every subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$, the operator system $\mathcal{S}_0$ defined in Eq. (17) is such that all the POVMs X with $X(\omega) \in \mathcal{S}_0$ for all ω are simultaneously correctable.*

Let us summarize how to obtain $\mathcal{S}_0$. We restricted the channel to the subspace $\mathcal{H}_0$, computed the correctable algebra $\mathcal{A}_0$ and the correction channel $\mathcal{R}_0$ for the restricted channel, and finally set $\mathcal{S}_0 = \mathcal{E}^*(\mathcal{R}_0^*(\mathcal{A}_0))$. In fact we can be entirely explicit. Letting V be the isometry embedding $\mathcal{H}_0$ into $\mathcal{H}_1$, $P := VV^\dagger$ the projector on $\mathcal{H}_0$, $\mathcal{E}_\lambda$ the regularized channel defined on the whole of $\mathcal{B}(\mathcal{H}_1)$, and

$$K := (\mathcal{E}_\lambda(P))^{-\frac{1}{2}}$$

we have that the operator system

$$\mathcal{S}_0 = \{\mathcal{E}^*(K\mathcal{E}_\lambda(VAV^\dagger)K) \ : A \in \mathcal{B}(\mathcal{H}_0),$$
$$[A, V^\dagger E_i^\dagger E_j V] = 0 \text{ for all } i, \, j\}$$

is corrected on $\mathcal{E}$ by the channel

$$\mathcal{R}(\rho) = P\mathcal{E}_\lambda^*(K\rho K)P.$$

For an explicit example in finite-dimension, see [7].

Do these structures exhaust all the correctable observables for $\mathcal{E}$? Theorem 3 of [9] states that, when $\mathcal{H}_1$ is finite-dimensional, the fixed point set of the dual of any channel must be made of elements of the form $A + \mathcal{F}^*(A)$ where A belongs to a $*$-algebra $\mathcal{A}_0$ inside $\mathcal{M}(\mathcal{H}_0)$, $\mathcal{H}_0$ a subspace of $\mathcal{H}_1$, and $\mathcal{F}^*$ is a fixed channel which is such that $P\mathcal{F}^*(A)P = 0$, where P projects onto $\mathcal{H}_0$.

If an operator system $\mathcal{S}$ is correctable for $\mathcal{E}$, then it is fixed by $\mathcal{E}^* \circ \mathcal{R}^*$ for some channel $\mathcal{R}$, and, according to [9], made of elements of the form mentioned above: $A + \mathcal{F}^*(A)$. This means that if V is the isometry embedding $\mathcal{H}_0$ into $\mathcal{H}_1$, we have $V^\dagger \mathcal{E}^*(\mathcal{R}^*(A + \mathcal{F}^*(A)))V = V^\dagger(A + \mathcal{F}^*(A))V = A$, which implies that the algebra $\mathcal{A}_0$ is correctable for $\mathcal{E}$ restricted to $\mathcal{H}_0$, and therefore belongs to the correctable algebra for this restricted channel. This shows that $\mathcal{S}$ is of the form covered by Theorem 4.2, which therefore exhausts all correctable observables on a finite dimensional system.

4.1. **Simultaneously correctable channels.** In Subsection 2.1 we motivated a situation where, of the channel elements E_i, only their span is known. It is already clear that the correctable algebra

$$\mathcal{A} = \{A \ : \ [A, E_i^\dagger E_j] = 0 \text{ for all } i, \, j\}$$

only depends on the span of the elements E_i. Indeed, consider a channel $\mathcal{E}$ with elements E_i and a channel $\mathcal{E}'$ with elements $F_i = \sum_j \gamma_{ij} E_j$ where γ_{ij} are arbitrary, provided that $\sum_i F_i^\dagger F_i = \mathbf{1}$. Then it is clear that an operator which commutes with the products $E_i^\dagger E_j$ for all i and j will also commute with operators $F_i^\dagger F_j$ since they are just linear combinations of the former. This fact tells us that $\mathcal{A}$ is the correctable algebra for all channels whose elements are chosen in the span of the operators E_i. However this fact alone would not be very helpful if the correction channel itself depended on the particular choice of channel elements. Fortunately, it does not.

In fact, we have already exploited part of this freedom in defining $\mathcal{R}$; recall that we defined it in terms of the map

$$\mathcal{E}_\lambda(A) = \sum_i \lambda_i E_i A E_i^\dagger$$

defined on $\mathcal{B}(\mathcal{H}_1)$ (see Eq. (12)). The sequence λ was chosen so that the infinite sum in the expression for $\mathcal{E}_\lambda$ is well defined for any operator. However, the exact value of the components λ_i did not matter in the proof that

$\mathcal{R}$ corrects the algebra $\mathcal{A}$ for the channel $\mathcal{E}$. The only important aspect of this channel was that its elements are linear combinations of the adjoints of the elements of $\mathcal{E}$.

The channel $\mathcal{R}'$ correcting $\mathcal{E}'$ would be defined in the same way in terms of the channel

$$\mathcal{E}'_\lambda(A) = \sum_i \lambda_i F_i A F_i^\dagger$$
$$= \sum_{ijk} \lambda_i \gamma_{ij} \overline{\gamma}_{ik} E_j A E_k^\dagger$$

which has also the right form for the corresponding correction channel

$$\mathcal{R}'(\rho) = (\mathcal{E}'_\lambda)^* ((\mathcal{E}'_\lambda(\mathbf{1}))^{-\frac{1}{2}} \rho (\mathcal{E}'_\lambda(\mathbf{1}))^{-\frac{1}{2}})$$

to correct the channel $\mathcal{E}$. We will not go through the proof that $\mathcal{R}'$ corrects $\mathcal{E}$ on $\mathcal{A}$, since precisely the same steps can be followed as for $\mathcal{R}$ itself.

Hence we have seen that all the channels whose elements span the same space of operators will have the same correctable algebra, and be correctable through the same correction channel. This means that this theory can be applied to the case described in Subsection 2.1, where the span of the elements is all that we know about the channel.

There is a sense in which it is this fact which allows for the quantum errors to be understood as being *discrete* [27]. Indeed, a standard error model for quantum computing is that where the system considered is a tensor product of *qubits*, namely two-dimensional quantum systems. The possible "errors" (i.e. possible channel elements of the noise) are supposed to be any operator acting on no more than n subsystems, where n is fixed. It is clear that this set of errors is continuous. However, for a finite number of qubits their span is separable (in fact finite-dimensional), which means that it suffices to choose a discrete set which spans the space and try to correct these only.

This discussion applies to simultaneously correctable sets of observables characterized by an algebra, which are the correctable sharp observables. However, it does not apply to the classes of simultaneously correctable unsharp observables identified in the previous section. Indeed, in those cases the correctable operator systems $\mathcal{S}_0$ may be different for two channels whose elements span the same operator space. To see this, remember that

$$\mathcal{S}_0 = \mathcal{E}^*(\mathcal{R}_0^*(\mathcal{A}_0)),$$

where $\mathcal{A}_0$ is the correctable algebra for the channel restricted to a subspace $\mathcal{H}_0$, and $\mathcal{R}_0$ the corresponding correction channel. Therefore, although both $\mathcal{R}_0$ and $\mathcal{A}_0$ would be the same for both channels, the set $\mathcal{S}_0$ in this expression depends explicitly on the action of the channel itself, and may be different in both cases.

4.2. **Nature of correctable channels.** The fact that $\mathcal{A}$ is the correctable algebra for the channel $\mathcal{E}$, with correction channel $\mathcal{R}$, implies that the map

$\mathcal{E}^* \circ \mathcal{R}^*$ acts simply as the identity on $\mathcal{A}$;

$$(\mathcal{E}^* \circ \mathcal{R}^*)|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$$

The following theorem is a direct generalization of the results presented in [8]. It elucidates what happens to the algebra $\mathcal{A}$ prior to its correction.

Theorem 4.3. *Let $\mathcal{A}$ be the correctable algebra for a channel $\mathcal{E}$. Then $\mathcal{E}^*$ is a normal $*$-homomorphism of the algebra generated by the pre-image of $\mathcal{A}$. In particular, for any operators B, B' such that $\mathcal{E}^*(B), \mathcal{E}^*(B') \in \mathcal{A}$, we have*

$$\mathcal{E}^*(BB') = \mathcal{E}^*(B)\mathcal{E}^*(B').$$

Proof. Remember that the projectors P in the correctable algebra $\mathcal{A}$ satisfy $BE_i = E_i P$ for some operator B, and for all i (see Eq. (10)). It can be directly checked that this is also true of the span of the projectors, which is the whole of the algebra $\mathcal{A}$, up to closure. Now consider two operators B and B' such that $A := \mathcal{E}^*(B)$ and $A' := \mathcal{E}^*(B')$ belong to the span of the projectors of $\mathcal{A}$. We know that $BE_i = E_i A$ and $B'E_i = E_i A'$. This implies that $BB'E_i = BE_i A' = E_i AA'$, from which it follows that

$$\mathcal{E}^*(BB') = \sum_i E_i^\dagger BB'E_i = \sum_i E_i^\dagger E_i AA' = AA' = \mathcal{E}^*(B)\mathcal{E}^*(B').$$

Since $\mathcal{E}^*$ is weak-$*$ continuous, this condition also applies to the weak-$*$ closure of the set of operators whose images are in the span of the projectors of $\mathcal{A}$. This shows that the above condition holds for every operator in the pre-image of $\mathcal{A}$. $\qquad\square$

Note that the pre-image of $\mathcal{A}$ under $\mathcal{E}^*$ includes in particular the image of the dual of any correction channel $\mathcal{R}$. In fact, the correction channel defined in Eq. (15) is itself a homomorphism. We saw in the above proof that for all operators A in the span of the projectors of $\mathcal{A}$, we have $\mathcal{R}^*(A)E_i = E_i A$. This implies that

$$\mathcal{R}^*(A)\mathcal{E}_\lambda(\mathbf{1}) = \mathcal{E}_\lambda(A),$$

which means explicitly

$$(\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}}\mathcal{E}_\lambda(A)(\mathcal{E}_\lambda(\mathbf{1}))^{\frac{1}{2}} = \mathcal{E}_\lambda(A),$$

or, simply,

$$[(\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}}, \mathcal{E}_\lambda(A)] = 0.$$

From the weak-$*$ continuity of $\mathcal{E}_\lambda$, this is true for all $A \in \mathcal{A}$. Using this fact, and also recalling that $[A, E_i^\dagger E_j] = 0$ for all $A \in \mathcal{A}$, yields

$$\begin{aligned}
\mathcal{R}^*(A)\mathcal{R}^*(A') &= (\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}}\mathcal{E}_\lambda(A)(\mathcal{E}_\lambda(\mathbf{1}))^{-1}\mathcal{E}_\lambda(A')(\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}} \\
&= (\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{3}{2}}\mathcal{E}_\lambda(A)\mathcal{E}_\lambda(A')(\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}} \\
&= (\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{3}{2}}\mathcal{E}_\lambda(\mathbf{1})\mathcal{E}_\lambda(AA')(\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}} \\
&= (\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}}\mathcal{E}_\lambda(AA')(\mathcal{E}_\lambda(\mathbf{1}))^{-\frac{1}{2}} \\
&= \mathcal{R}^*(AA').
\end{aligned}$$

Hence, we have proved the following result.

Proposition 4.4. *The correction channel given by Eq. (15) is a faithful representation of the correctable von Neumann algebra $\mathcal{A}$.*

This shows that the effect of a channel on its correctable algebra simply amounts to representing it in a different way on the Hilbert space. These results clarify certain aspects of [25]. We refer to [8] for the precise relationship between our results and those of [25].

5. Algebraic Codes

In the rest of the paper we shall describe a number of special cases and examples that arise in our setting.

In Subsection 2.1 we mentioned that the purpose of quantum error correction was to find a "code" on which the channel can be inverted, without defining what we meant by a code. In the previous section, we started from the general assumption that a code should be a set of simultaneously correctable observables. We have then found that for any subspace $\mathcal{H}_0$ of the source Hilbert space $\mathcal{H}_1$, there is a set of simultaneously correctable observables characterized by an operator system $\mathcal{S}_0$, or equivalently by the von Neumann algebra $\mathcal{A}_0$ which is such that $\mathcal{A}_0 = V^\dagger \mathcal{S}_0 V$ where V is the isometry embedding $\mathcal{H}_0$ into $\mathcal{H}_1$. The algebra $\mathcal{A}_0$ characterizes the sharp observables correctable for the channel restricted to the subspace $\mathcal{H}_0$.

Hence, all the codes that we identified, on which the channel can be inverted, are, or correspond to, von Neumann algebras. In fact it is easy to build abstract examples which yield any possible von Neumann algebra in this way, given that all von Neumann algebras can be realized as the commutant of a set of operators [14].

The general structure of von Neumann algebras is well studied. Let us briefly review the basics here in order to understand the type of information that they represent.

5.1. **Structure of von Neumann algebras.** Let us first summarize the representation theory of finite-dimensional von Neumann algebras, which are just $*$-algebras.

A concrete finite-dimensional $*$-algebra $\mathcal{A}$, represented by matrices; i.e., operators on a finite-dimensional Hilbert space represented in a fixed orthonormal basis, always has the form

$$(19) \qquad \mathcal{A} = \bigoplus_{k=1}^{N} \mathcal{M}_{n_k} \otimes \mathbf{1}_{m_k}$$

where $\mathcal{M}_{n_k}$ denotes the full set of square matrices of size n_k and $\mathbf{1}_{m_k}$ the identity matrix of size m_k. If the dimension of the algebra $\mathcal{A}$ is D then we have $D = \sum_k n_k^2$. The direct sum of two matrix algebras must be understood as the algebra of block-diagonal matrices, with one block encoding the first

algebra, and the other block the second algebra. Therefore the above means that, written as a matrix of blocks,

$$\mathcal{A} = \begin{pmatrix} \mathcal{M}_{n_1} \otimes \mathbf{1}_{m_1} & 0 & \cdots & 0 \\ 0 & \mathcal{M}_{n_2} \otimes \mathbf{1}_{m_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{M}_{n_N} \otimes \mathbf{1}_{m_N} \end{pmatrix}.$$

In addition, tensoring a matrix algebra with the identity on another algebra means that we are considering matrices which are also block-diagonal, with as many blocks as there are elements on the diagonal of the identity matrix, but such that all blocks are identical, not only in their size, but also in their content.

For instance, any operator A in the algebra $\mathcal{A} = (\mathcal{M}_2 \otimes \mathbf{1}_2) \oplus (\mathcal{M}_3)$ has the form

$$A = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

for a two-by-two matrix B and a three-by-three matrix C.

The block-diagonal structure of $\mathcal{A}$ is determined by the form of its *center* $\mathcal{Z}(\mathcal{A})$. The center is the set of operators inside the algebra which commute with all other elements of the algebra;

$$\mathcal{Z}(\mathcal{A}) = \{ A \in \mathcal{A} \ : \ [A, B] = 0 \text{ for all } B \in \mathcal{A} \ \}.$$

It is itself a commutative von Neumann algebra. The center can also be written as the intersection of the algebra with its *commutant* $\mathcal{A}'$ which is the algebra composed of all operators commuting with all elements of $\mathcal{A}$;

$$\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'.$$

For instance, for an algebra of the form $\mathcal{M}_n \otimes \mathbf{1}_m$, we have

$$\mathcal{Z}(\mathcal{M}_n \otimes \mathbf{1}_m) = (\mathcal{M}_n \otimes \mathbf{1}_m) \cap (\mathbf{1}_n \otimes \mathcal{M}_m) \approx \mathbb{C}.$$

A von Neumann algebra is said to be a *factor* if its center is isomorphic to $\mathbb{C}$. Hence matrix algebras of the form $\mathcal{M}_n \otimes \mathbf{1}_m$ are factors.

More generally, if the representation of $\mathcal{A}$ is expressed as in Eq. (19), then the commutant is

$$\mathcal{A}' = \bigoplus_{k=1}^{N} \mathbf{1}_{n_k} \otimes \mathcal{M}_{m_k}$$

and the center of $\mathcal{A}$ is

(20) $$\mathcal{Z}(\mathcal{A}) = \bigoplus_{k=1}^{N} \mathbb{C}(\mathbf{1}_{n_k} \otimes \mathbf{1}_{m_k})$$

which means that it is composed of diagonal matrices with only N different eigenvalues. If P_k is the projector on the kth block, then this means that a

generic element $C \in \mathcal{Z}(\mathcal{A})$ of the center is of the form

$$C = \sum_k c_k P_k$$

for arbitrary complex numbers c_k. The algebra $\mathcal{A}$ itself is block-diagonal in terms of the subspaces defines by the projectors P_k, in the sense that for all $A \in \mathcal{A}$,

$$A = \sum_i P_k A P_k.$$

Hence the center of the algebra essentially tells us what the blocks are in its representation.

For instance, consider again the algebra $\mathcal{A} = (\mathcal{M}_2 \otimes \mathbf{1}_2) \oplus (\mathcal{M}_3)$. Typical operators $A' \in \mathcal{A}'$ and $C \in \mathcal{A}$ have the form

$$A' = \begin{pmatrix} a\mathbf{1}_2 & b\mathbf{1}_2 & 0 \\ c\mathbf{1}_2 & d\mathbf{1}_2 & 0 \\ 0 & 0 & x\mathbf{1}_3 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a\mathbf{1}_2 & 0 & 0 \\ 0 & a\mathbf{1}_2 & 0 \\ 0 & 0 & x\mathbf{1}_3 \end{pmatrix},$$

where $a, b, c, d, x \in \mathbb{C}$.

When $\mathcal{A}$ is infinite-dimensional, the direct sum must be replaced by a *direct integral*. This follows from the fact that the center can be any commutative algebra, which has the form

$$\mathcal{Z}(\mathcal{A}) \approx L^\infty(\Omega)$$

for some set Ω equipped with a measure. It is with respect to this measure that we can write

$$\mathcal{A} \approx \int_\Omega^\oplus \mathcal{A}(x) \, dx$$

Where the generalized "blocks" $\mathcal{A}(x)$ are factors; i.e., have a trivial center. If Ω is finite then we must use a discrete measure, which gives us the direct sum in Eq. (20). Hence this integral can be intuitively understood as a continuous limit of the direct sum.

Factors come in three main types. Up to now we have been using *type I factors*, which can always be represented as $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. They are characterized abstractly by the fact that some of the projections they contain are *minimal*, which means that there is no smaller nonzero projection in the algebra. When the algebra is represented as $\mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$, the minimal projections are the projectors on local pure states, i.e. of the form $P = |\psi\rangle\langle\psi| \otimes \mathbf{1}$.

The factors that have no minimal projections are further classified into types II and III according to other properties of their projections. Type II factors have been extensively studied in pure mathematics. Below we shall discuss a specific instance of a type II factor that arises in our setting. Type III factors are of central interest in physics since they describe local algebras of observables in relativistic quantum field theory [18]. In this sense they are the proper way to model local degrees of freedom.

5.2. **Subspace codes and subsystem codes.** Traditionally, a *quantum error correcting code* is just a subspace $\mathcal{H}_0$ of the initial Hilbert space $\mathcal{H}_1$, which is assumed to be finite-dimensional [4, 30, 31, 16, 20, 27]. The idea is that the channel $\mathcal{E}$ is correctable for states in $\mathcal{H}_0$ if there is a channel $\mathcal{R}$ such that

$$\mathcal{R}(\mathcal{E}(\rho)) = \rho,$$

for all states ρ which are mixtures of pure states in the subspace $\mathcal{H}_0$. If we introduce the isometry $V : \mathcal{H}_0 \to \mathcal{H}_1$ which embeds $\mathcal{H}_0$ into $\mathcal{H}_1$, this means that

$$\mathcal{R}(\mathcal{E}(V\rho V^\dagger)) = V\rho V^\dagger$$

for all $\rho \in \mathcal{B}_t(\mathcal{H}_0)$, which is equivalent to requiring the existence of a channel $\mathcal{R}'$ such that

$$\mathcal{R}'(\mathcal{E}(V\rho V^\dagger)) = \rho$$

for all $\rho \in \mathcal{B}_t(\mathcal{H}_0)$. Indeed, it suffices to pick $\mathcal{R}'(\rho) = V^\dagger \mathcal{R}(\rho)V$.

If we define $\mathcal{E}_0(\rho) := \mathcal{E}(V\rho V^\dagger)$, this means that $\mathcal{R}' \circ \mathcal{E}_0$ is the identity on $\mathcal{B}_t(\mathcal{H}_0)$, or equivalently that $\mathcal{E}^* \circ (\mathcal{R}')^*$ is the identity on $\mathcal{B}(\mathcal{H}_0)$, which, as we have shown matches our conception of correctability for the algebra $\mathcal{B}(\mathcal{H}_0)$. Therefore we recover the framework of standard (subspace) quantum error correction, for a code $\mathcal{H}_0$, when the correctable algebra is $\mathcal{B}(\mathcal{H}_0)$, and the channel is restricted to the subspace $\mathcal{H}_0$.

In order to complete the comparison, let us check that our correctability condition reduces to the one introduced for standard codes [20]. The Knill-Laflamme condition states that a standard code represented by the subspace $\mathcal{H}_0$ is correctable for the channel $\mathcal{E}$ with elements E_i if and only if there exists scalars $\lambda_{ij} \in \mathbb{C}$ such that

$$(21) \qquad\qquad V^\dagger E_i^\dagger E_j V = \lambda_{ij}\mathbf{1} \quad \forall i, j,$$

where V embeds $\mathcal{H}_0$ into the source Hilbert space $\mathcal{H}_1$. In our framework, the correctable algebra must be precisely the commutant of the operators $V^\dagger E_i^\dagger E_j V$ for all i and j. Since here they are all proportional to the identity on $\mathcal{H}_0$, the correctable algebra is indeed the whole algebra of operators on $\mathcal{H}_0$.

A more general framework was also introduced which generalized the notion of a code to that of a *subsystem code* [23, 24]. In this approach one defines a code through a subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$ (where $\mathcal{H}_1$ is still finite-dimensional) and a particular subsystem decomposition $\mathcal{H}_A \otimes \mathcal{H}_B = \mathcal{H}_0$ of this subspace. Again, let V be the isometry embedding $\mathcal{H}_0$ into $\mathcal{H}_1$. We then say that the subsystem $\mathcal{H}_A$ is a correctable code if there is a channel $\mathcal{R}$ such that

$$\mathcal{R}(\mathcal{E}(V(\rho \otimes \tau)V^\dagger)) = \rho \otimes \tau'$$

for any states $\rho \in \mathcal{B}_t(\mathcal{H}_A)$, $\tau, \tau' \in \mathcal{B}_t(\mathcal{H}_B)$. We want to show that this is equivalent to the case where the correctable algebra $\mathcal{A}$, in our framework, is any finite-dimensional factor of type I, which in this case is

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathbf{1}_B.$$

That is, assuming we are restricting the initial state to the subspace $\mathcal{H}_0$. In our language, this would mean that

$$(22) \qquad V^\dagger \mathcal{E}^*(\mathcal{R}^*(X \otimes \mathbf{1}))V = X \otimes \mathbf{1}$$

for all $X \in \mathcal{B}(\mathcal{H}_A)$. Indeed, suppose first that $\mathcal{H}_A$ is a subsystem code corrected by $\mathcal{R}$, then we have for all $X \in \mathcal{B}(\mathcal{H}_A)$,

$$\begin{aligned}
\mathrm{Tr}(V^\dagger \mathcal{E}^*(\mathcal{R}^*(X \otimes \mathbf{1}))V(\rho \otimes \sigma)) &= \mathrm{Tr}((X \otimes \mathbf{1})\mathcal{R}(\mathcal{E}(V\rho \otimes \sigma V^\dagger))) \\
&= \mathrm{Tr}(X\rho \otimes \tau) = \mathrm{Tr}(X\rho)\mathrm{Tr}(\tau) \\
&= \mathrm{Tr}(X\rho) = \mathrm{Tr}((X \otimes \mathbf{1})(\rho \otimes \sigma)).
\end{aligned}$$

This is true for all states $\rho \in B_t(\mathcal{H}_A)$ and all states $\sigma \in \mathcal{B}_t(\mathcal{H}_B)$. By linearity it follows that $V^\dagger \mathcal{E}^*(X \otimes \mathbf{1})V = X \otimes \mathbf{1}$ for all $X \in B(\mathcal{H}_A)$. Conversely, if Eq. (22) is true for all X, then for all $\rho \in B_t(\mathcal{H}_0)$ we have

$$\begin{aligned}
\mathrm{Tr}(X\, \mathrm{Tr}_B(\mathcal{R}(\mathcal{E}(V\rho V^\dagger)))) &= \mathrm{Tr}((X \otimes \mathbf{1})\mathcal{R}(\mathcal{E}(V\rho V^\dagger))) \\
&= \mathrm{Tr}(V^\dagger \mathcal{E}^*(\mathcal{R}^*(X \otimes \mathbf{1}))V\rho) \\
&= \mathrm{Tr}((X \otimes \mathbf{1})\rho) \\
&= \mathrm{Tr}(X\, \mathrm{Tr}_B(\rho)).
\end{aligned}$$

Since the above equation is true for all X, we have $\mathrm{Tr}_B(\mathcal{R}(\mathcal{E}(V\rho V^\dagger))) = \mathrm{Tr}_B(\rho)$ for all $\rho \in \mathcal{B}_t(\mathcal{H}_0)$, which was shown in [24] to be equivalent to the definition of $\mathcal{H}_A$ being a noiseless subsystem for $\mathcal{E}$.

In this framework, the correctability condition reads [23]

$$(23) \qquad V^\dagger E_i^\dagger E_j V = \mathbf{1} \otimes \Lambda_{ij}$$

for an arbitrary set of operators $\Lambda_{ij} \in \mathcal{B}(\mathcal{H}_B)$. This means that the operators $V^\dagger E_i^\dagger E_j V$ for all i and j generate the sub-algebra $\mathbf{1} \otimes \mathcal{B}(\mathcal{H}_B)$ of $\mathcal{B}(\mathcal{H}_0)$, whose commutant is indeed $\mathcal{B}(\mathcal{H}_A) \otimes \mathbf{1}$; the correctable algebra defining the subsystem code.

Note that every subsystem code is associated with a family of standard codes, whenever we can afford to put stronger constraints on the initial state. Indeed, consider the smaller subspace $\mathcal{H}_0'$ formed by the states inside $\mathcal{H}_0$ which are of the form $|\psi\rangle \otimes |\phi_0\rangle$, where $|\phi_0\rangle$ is fixed. This subspace is associated with the isometry $W = V \otimes |\phi_0\rangle$, for which we have

$$W^\dagger E_i^\dagger E_j W = \langle\phi_0|\Lambda_{ij}|\phi_0\rangle \mathbf{1}$$

which is just the Knill-Laflamme condition for $\mathcal{H}_0'$.

These results show that the standard subspace codes, as well as the subsystem codes, correspond to the case where our correctable algebra is a finite-dimensional factor, which is always of type I. Our results yield several types of generalizations over these codes. Firstly, we obtain a characterization of infinite-dimensional quantum codes, which will be discussed further in Subsection 5.4 and Sections 6 and 7. We also obtain a characterization of discrete or continuous classical codes, corresponding to a commutative correctable algebra. This case will not be discussed here. In addition, we can

handle the correction of information that is neither quantum nor classical; i.e., that is represented by an algebra which is neither commutative nor a factor.

5.3. **Hybrid classical-quantum codes.** We have seen in our framework that the structure to be corrected can be any von Neumann algebra. A general von Neumann algebra with center $\mathcal{Z}(\mathcal{A}) = L^\infty(\Omega)$ is of the form

$$(24) \qquad \mathcal{A} = \int_\Omega^\oplus \mathcal{A}(x)dx$$

where each $\mathcal{A}(x)$ is a factor. If the center is maximal, $\mathcal{Z}(\mathcal{A}) = \mathcal{A}$, then the algebra $\mathcal{A}$ is commutative and each factor $\mathcal{A}(x)$ is of dimension one, and hence isomorphic to the complex numbers $\mathbb{C}$. If, on the other hand, the center is minimal, $\mathcal{Z}(\mathcal{A}) \approx \mathbb{C}$, then the set Ω contains only a single element x_0 and $\mathcal{A} = \mathcal{A}(x_0)$ is a factor.

If $\mathcal{A}$ is commutative, then it represents a classical system, which is clear from the fact that it has the form $L^\infty(\Omega)$. It is then natural to say that if it is a factor, it represents a "pure" quantum system.

A physical system represented by an algebra, whose structure is given by the general form represented in Eq. (24), can be understood as being partly quantum and partly classical. Indeed, we can consider the center $\mathcal{Z}(\mathcal{A}) = L^\infty(\Omega)$ as representing a classical system. For each possible "state" $x \in \Omega$ of this classical system, we have a pure quantum system represented by the factor $\mathcal{A}(x)$.

For instance, a classical system, represented by $L^\infty(\Omega)$, next to a type I quantum system, with algebra $\mathcal{B}(\mathcal{H})$, is represented by

$$\mathcal{A} = L^\infty(\Omega) \otimes \mathcal{B}(\mathcal{H}) \simeq \int_\Omega^\oplus \mathcal{A}(x)dx$$

where each factor $\mathcal{A}(x)$ is a copy of $\mathcal{B}(\mathcal{H})$. In the more general case, however, the size and type of the algebra $\mathcal{A}(x)$ may depend upon x.

Generic operators $A, B \in \int_\Omega^\oplus \mathcal{A}(x)\, dx$ are of the form

$$(25) \qquad A = \int_\Omega A_x\, dx \qquad \text{and} \qquad B = \int_\Omega B_x\, dx,$$

where $A_x, B_x \in \mathcal{A}(x)$ for all $x \in \Omega$. Their product is simply

$$AB = \int_\Omega A_x B_x\, dx.$$

An element of the center is of the form

$$C = \int_\Omega \alpha(x)\mathbf{1}_x\, dx$$

where $\alpha \in L^\infty(\Omega)$ and $\mathbf{1}_x$ is the identity on $\mathcal{A}_x$.

If A in Eq. (25) is an effect, then each operator A_x is also an effect that can be interpreted as a quantum proposition which is true conditionally on the classical system being in state x.

If the finite-dimensional case, if a hybrid algebra

$$\mathcal{A} = \bigoplus_k \mathcal{M}_{n_k} \otimes \mathbf{1}_{m_k}$$

is correctable, then each factor $\mathcal{A}_k = \mathcal{M}_{n_k} \otimes \mathbf{1}_{m_k}$ represents a correctable subsystem code for states restricted to the subspace $\mathcal{H}_k$ projected onto by $P_k = \mathbf{1}_{n_k} \otimes \mathbf{1}_{m_k}$. Indeed, if V is the isometry corresponding to P_k, then $VAV^\dagger$ is the representation inside $\mathcal{A}$ of an operator $A \in \mathcal{A}_k$. Thus

$$V^\dagger \mathcal{E}^*(\mathcal{R}^*(VAV^\dagger))V = V^\dagger VAV^\dagger V = A.$$

Hence a finite-dimensional hybrid algebra can be understood as representing a family of orthogonal subsystem codes correctable simultaneously.

5.4. **Infinite-dimensional subspace and subsystem codes.** The derivation of necessary and sufficient conditions for error correction of infinite-dimensional algebras is a new result which is potentially significant, given that all physical systems are naturally modelled by infinite-dimensional systems. In particular, it yields a formulation of quantum error correction for systems characterized by continuous variables [11].

As we discussed above, a code can be said to be purely quantum if it is represented by an algebra which is a factor. In the finite-dimensional case, we have seen in Subsection 5.2 that factors represent subsystem codes characterized by Eq. (23), or Eq. (21). Some authors assumed that this condition would hold unchanged in the infinite-dimensional case. For instance, in [10] the Knill-Laflamme condition was expressed for a channel with continuous elements E_x as

$$V^\dagger E_x^\dagger E_y V = \lambda(x, y)\mathbf{1}.$$

where $\lambda(x, y) \in \mathbb{C}$. Our results show that this condition is in fact *sufficient*. Indeed, it implies that the commutant of the operators $V^\dagger E_x^\dagger E_y V$ for all x and y is the whole algebra $\mathcal{B}(\mathcal{H}_0)$ on the subspace $\mathcal{H}_0 \subseteq \mathcal{H}_1$. However, in the infinite-dimensional case this condition is no longer necessary since it expresses that the code must be isomorphic to $\mathcal{B}(\mathcal{H}_0)$, which is a factor of type I. Hence it misses the possibility of correcting more general factors. In particular, our generalization of quantum error correction to infinite-dimensional systems introduces new types of quantum codes not previously considered, namely infinite-dimensional type I subsystem codes, factors of type II, and factors of type III. An example of a correctable type II factor is given in Section 7 below.

In the next section we illustrate our framework with an elementary example of an infinite-dimensional type I factor code.

6. Type I Infinite-Dimensional Example

Here we consider what is arguably the simplest class of bona fide infinite-dimensional quantum error correcting subspace codes.

Let $\mathcal{H}_0$ be a subspace of an infinite-dimensional Hilbert space $\mathcal{H}$. The only constraint we place on the dimensionality of $\mathcal{H}_0$ is that it have infinite co-dimension (in particular it could be infinite-dimensional itself).

Let P be the projection of $\mathcal{H}$ onto $\mathcal{H}_0$, and let $\{P_i\}_{i=1}^{\infty}$ be a family of projections of $\mathcal{H}$ onto mutually orthogonal subspaces $\mathcal{H}_i$, each of the same dimension as $\mathcal{H}_0$. Further let $\{p_i\}_{i=1}^{\infty}$ satisfy $p_i > 0$ and $\sum_{i \geq 1} p_i = 1$, and let $\{V_i\}_{i=1}^{\infty}$ be a family of isometries with $\mathcal{H}_0$ as their common initial space such that V_i maps $\mathcal{H}_0$ isometrically onto $\mathcal{H}_i$. This is equivalent to the following operator equations:

$$P_i P_j = \delta_{ij} P_i, \quad V_i^{\dagger} V_i = \mathbf{1}_{\mathcal{H}_0}, \quad V_i V_i^{\dagger} = P_i, \quad V_i = P_i V_i, \quad \forall i, j.$$

It follows that $\mathcal{H}_0$ is entirely correctable for any channel $\mathcal{E}$ which, when restricted to $\mathcal{H}_0$, acts as

$$\mathcal{E}_0(\rho) = \sum_{i=1}^{\infty} p_i \, V_i \rho V_i^{\dagger}.$$

Indeed, since $V_i^{\dagger} V_j = \delta_{ij} \mathbf{1}_{\mathcal{H}_0}$, the commutant of the set $\{V_i^{\dagger} V_j\}_{ij}$ is the whole of $\mathcal{B}(\mathcal{H}_0)$. This class of channels and codes is a straightforward generalization of the class that plays a central role in the proof of the Knill-Laflamme characterization [20] of subspace codes for the finite-dimensional case.

For the sake of the example let us compute the correction channel by directly applying Eq. (15). First note that $\mathcal{E}_0$ is well defined on any bounded operator, therefore we can use it in place of the modified channel $\mathcal{E}_\lambda$ introduced in Eq. 12. We have

$$(\mathcal{E}_0(\mathbf{1}))^{-\frac{1}{2}} = \sum_i \frac{1}{\sqrt{p_i}} P_i,$$

so that the correction channel is

$$\mathcal{R}(\rho) = \sum_{ijk} p_i V_i^{\dagger} \frac{1}{\sqrt{p_j}} P_j \rho P_k \frac{1}{\sqrt{p_k}} V_i = \sum_i V_i^{\dagger} \rho V_i.$$

This illustrates how the correction channel becomes independent of the particular probabilities p_i.

7. Type II Example: Irrational Rotation Algebra

Consider the algebra generated by two elements $\hat{x}$ and $\hat{p}$ satisfying the canonical commutation relations

$$[\hat{x}, \hat{p}] = i\mathbf{1}.$$

This algebra can be represented on $\mathcal{H} = L^2(\mathbb{R})$, where the position operator $\hat{x}$ acts on a function $\psi \in L^2(\mathbb{R})$ as $(\hat{x}\psi)(x) = x\psi(x)$ and the momentum $\hat{p}$ as $(\hat{p}\psi)(x) = i\frac{d}{dx}\psi(x)$.

Suppose that this system interacts with an environment through a Hamiltonian of the form $H = \sum_i J_i \otimes K_i$, where the operators J_i act on the system,

and the operators K_i on the environment. We further assume that the interaction operators J_i are of two forms: some are periodic functions of $\hat{x}$, with period L_x, and others are periodic functions of $\hat{p}$, of period L_p. This implies that these functions are linear combinations of powers of the functions $x \mapsto e^{i\frac{2\pi}{L_x}x}$ or $p \mapsto e^{i\frac{2\pi}{L_p}p}$ respectively (their discrete Fourier components). For convenience, let us define

$$\omega_x := \frac{2\pi}{L_x} \quad \text{and} \quad \omega_p := \frac{2\pi}{L_p}.$$

Since the interaction operators are bounded, we can follow the reasoning presented in Subsection 2.1 and conclude that the channel elements of the resulting channel on the system belong to the algebra generated by the operators J_i.

The von Neumann algebra generated by the interaction operators is also generated by the two unitary operators

$$U = e^{i\omega_x \hat{x}} \quad \text{and} \quad V = e^{i\omega_p \hat{p}}.$$

These can be understood as the two possible errors in our noise model.

To make things more interesting, we assume that the real number

$$\theta := \frac{\omega_x \omega_p}{2\pi}$$

is irrational. This number is important because it enters into the commutator of U and V:

$$UV = e^{2\pi i\theta}VU.$$

For this example, we will look for codes which are correctable for any initial state. In principle, in order to find the correctable algebra, we need to find the operators commuting with the products $E_i^\dagger E_j$ of the channel elements E_i. However, since we only know the span of these operators, we cannot exclude that $E_i = \mathbf{1}$ for some i. If this is the case, then these products include E_i and $E_i^\dagger$ for all i. Therefore, for an effect to be correctable, it needs to be in the commutant of the von Neumann algebra generated by the operators E_i for all i, which is the same as the von Neumann algebra generated by the interaction operators J_i, or simply U and V.

The operators

$$U' = e^{i\frac{\omega_x}{\theta}\hat{x}} \quad \text{and} \quad V' = e^{i\frac{\omega_p}{\theta}\hat{p}}$$

commute with both U and V. To see that U' commutes with V, simply note that

$$U'V = e^{i\frac{\omega_x \omega_p}{\theta}}VU' = e^{i\frac{2\pi\theta}{\theta}}VU' = VU'.$$

Similarly, V' also commutes with both U and V. In fact, the von Neumann algebra generated by U' and V' is the whole commutant of the algebra generated by the interaction operators J_i. In addition, it happens to be a factor of type II, and, together, with its commutant they generate the whole of $\mathcal{B}(\mathcal{H})$ [15].

Therefore, this is an example of a correctable factor of type II. In fact, it is also *noiseless* [21], in the sense that the correction channel can be taken to be the identity channel; i.e., no active correction is needed. This happens simply because $\mathbf{1}$ was assumed to be among the channel elements. Indeed, we saw that it implied that the correction operators had to commute with the channel elements themselves. Hence, if $\mathcal{E}$ is the channel, we have $\mathcal{E}^*(A) = \mathcal{E}^*(\mathbf{1})A = A$ for all elements A of the correctable algebra.

Let us see how we can understand this "type II subsystem", and how it resembles, and differs from, the factors of type I with which we are familiar.

If we were dealing with a factor of type I containing the identity, then the Hilbert space would take the form $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, so that our algebra would be simply $\mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}$. For instance, consider $\mathcal{H} = L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. If $\psi \in L^2(\mathbb{R}^2)$, the operators in the first factor $\mathcal{A} = \mathcal{B}(L^2(\mathbb{R})) \otimes \mathbf{1}$ are those which act only on the first component of ψ. The first factor is generated by the operators $(\hat{x}\psi)(x,y) = x\psi(x,y)$ and $(\hat{p}\psi)(x,y) = \frac{d}{dx}\psi(x,y)$. Note that here the set $\mathbb{R}^2$ on which the states are defined can be understood to be the set of joint eigenvalues to the position operators in $\mathcal{A}$ and $\mathcal{A}'$.

Something similar happens for our factor of type II. Let $\mathcal{A}$ be the factor generated by U and V, and $\mathcal{A}'$ its commutant, which is generated by U' and V'. Let us see if we could see the elements of $\mathcal{H}$ as wavefunctions over the eigenvalues of U and U'. First, note that the spectrum of both these operators is the unit circle in the complex plane. These two operators being functions of the position operator $\hat{x}$, we may wish to use the fact that the states of $\mathcal{H}$ can be represented as wavefunctions over the spectrum of $\hat{x}$, that is elements of $L^2(\mathbb{R})$. Indeed, we can naturally convert an eigenvalue x of $\hat{x}$ into the eigenvalues

$$(26) \qquad a = e^{i\omega_x x} \quad \text{and} \quad b = e^{i\frac{\omega_x}{\theta}x}$$

respectively of U and U'. In fact, this relationship is invertible. Indeed, if we are given a and b, then only a single real number x will satisfy both these relations. Indeed, suppose that we had two different real numbers x and x' yielding the same values of a and b. This would imply that they are related by $x - x' = 2\pi n/\omega = 2\pi m\theta/\omega$ for two integers n and m. But this would imply $\theta = n/m$, which is not possible since we assumed θ to be irrational. This implies that for each state $\psi \in L^2(\mathbb{R})$, we can define the function

$$\widetilde{\psi}(a,b) := \psi(x)$$

where x is the unique real number related to a and b via Eq. (26). Note that this function $\widetilde{\psi}$ is defined only on the valid couples (a,b) related to some $x \in \mathbb{R}$ via Eq. 26. However, due to the irrationality of θ, these couples are dense in the unit torus. We can therefore think of $\widetilde{\psi}$ as being defined almost everywhere.

We will see that $\widetilde{\psi}$ can be interpreted, in a suitable sense, as the wavefunction of a particle on a two-dimensional torus.

The relation between the wavefunctions $\widetilde{\psi}(a, b)$ and the factors $\mathcal{A}$ and $\mathcal{A}'$ is given by the fact that they "act" respectively on the first and second arguments of ψ respectively. Specifically, observe that

$$(U\widetilde{\psi})(a, b) = (U\psi)(x) = e^{i\omega_x x}\psi(x) = a\,\widetilde{\psi}(a, b),$$

which means that U acts just like the first component of the position of the particle. Similarly,

$$(U'\widetilde{\psi})(a, b) = (U'\psi)(x) = e^{i\frac{\omega_x}{\theta}x}\psi(x) = b\,\widetilde{\psi}(a, b).$$

The action of V is also easy to compute,

$$(V\widetilde{\psi})(a, b) = (V\psi)(x) = \psi(x + \omega_p) = \psi(a\,e^{2\pi i\theta}, b).$$

Hence, its action is to rotate the first argument by the irrational angle θ. Similarly,

$$(V'\widetilde{\psi})(a, b) = (V'\psi)(x) = \psi(x + \omega_p/\theta) = \psi(a, b\,e^{2\pi i\frac{1}{\theta}}).$$

Although it looks like a particle on the torus, this system differs from it by the nature of the normalized states. Indeed, the norm is

$$\|\widetilde{\psi}\|^2 = \int |\psi(x)|^2\,dx = \int |\widetilde{\psi}(e^{i\omega_x x}, e^{i\frac{\omega_x}{\theta}x})|^2\,dx,$$

and so what we have done is to take a standard particle in a one-dimensional space, and wrap its space around a torus in a dense trajectory. If we view the particle as a wavefunction $\widetilde{\psi}$ on the torus, its norm is an integral over this path which is dense in the torus.

This picture illustrates what the noise does. It disturbs only the first component of the position of this particle, but not the second.

8. Outlook

There are several significant lines of investigation that are suggested by this work. One is to use the theory presented here as the basis to extend more of the results of quantum error correction to the infinite-dimensional setting. While some results and techniques will surely extend straightforwardly, the present results indicate a range of qualitatively new phenomena and correspondingly new possibilities for quantum error correction in the infinite-dimensional case, such as correctable type II and type III factors. Type III factors, in particular, naturally describe local algebras of observables in relativistic quantum field theory [18]. We close by briefly touching on three specific potential avenues of research.

First, in this work we have assumed that the noise model was given by a quantum channel specified by the span of its elements, E_k. Alternatively, a noise model can be specified by giving the interaction Hamiltonian. The

link between the two descriptions (described for the case of bounded interaction operators in Subsection 2.1), becomes nontrivial in the infinite-dimensional setting. For instance, whereas the unitary time evolution operator is bounded, the interaction operators are generally unbounded and have a restricted domain and range.

Second, we note that although we have characterized a large class of simultaneously correctable unsharp observables (POVM) (Section 4), it is not clear as yet whether or not this exhausts all possible correctable observables. Previous results [9] indicate that there are no other correctable observables in finite-dimensional Hilbert spaces, but these results need to be generalized to the infinite-dimensional setting.

Finally, it should be interesting and relatively straightforward to extend our results to more general channels. For instance, it would be natural to consider normal completely positive maps defined between arbitrary von Neumann algebras, rather than just type I factors.

In conclusion, there are clearly deep connections between quantum error correction on the one hand and the rich field of operator algebras and operator systems on the other, leading to new possibilities for fruitful interactions between the two disciplines.

References

[1] P. Aliferis and A. W. Cross. Subsystem fault tolerance with the bacon-shor code. *Phys. Rev. Lett.*, 98:220502, 2007, arXiv:quant-ph/0610063.

[2] S. A. Aly and A. Klappenecker. Subsystem code constructions. 2007, arXiv:0712.4321.

[3] D. Bacon. Operator quantum error-correcting subsystems for self-correcting quantum memories. *Phys. Rev. A*, 73:012340, 2006.

[4] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters. Mixed-state entanglement and quantum error correction. *Phys. Rev. A*, 54:3824–3851, 1996.

[5] C. Beny. *Information flow at the quantum-classical boundary.* PhD in Applied Mathematics, Department of Applied Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, 2008.

[6] C. Beny, A. Kempf, and D. W. Kribs. Generalization of quantum error correction via the Heisenberg picture. *Phys. Rev. Lett.*, 98(10):100502, 2007.

[7] C. Beny, A. Kempf, and D. W. Kribs. Quantum error correction of observables. *Phys. Rev. A*, 76:042303, 2007.

[8] C. Beny, D. W. Kribs, and A. Pasieka. Algebraic formulation of quantum error correction. *International Journal of Quantum Information*, 6(1):597–603, 2008.

[9] R. Blume-Kohout, H. K. Ng, D. Poulin, and L. Viola. Characterizing the structure of preserved information in quantum processes. *Physical Review Letters*, 100(3):030501, 2008.

[10] S. L. Braunstein. Error correction for continuous quantum variables. *Phys. Rev. Lett.*, 80(18):4084–4087, 1998.

[11] S. L. Braunstein and A. K. Pati. *Quantum information with continuous variables*. Kluwer, Dordrecht, 2003.

[12] S. L. Braunstein and P. van Loock. Quantum information with continuous variables. *Reviews of Modern Physics*, 77(2):513, 2005.

[13] M.-D. Choi and D. W. Kribs. Method to find quantum noiseless subsystems. *Phys. Rev. Lett.*, 96:050501, 2006, arXiv:quant-ph/0507213.

[14] K. R. Davidson. *C*-algebras by example*. Amer. Math. Soc., Providence, 1996.

[15] L. Faddeev. Discrete Heisenberg-Weyl group and modular group. *Letters in Mathematical Physics*, 34:249, 1995.

[16] D. Gottesman. Class of quantum error-correcting codes saturating the quantum hamming bound. *Phys. Rev. A*, 54(3):1862–1868, 1996.

[17] D. Gottesman, A. Kitaev, and J. Preskill. Encoding a qubit in an oscillator. *Phys. Rev. A*, 64(1):012310, 2001.

[18] R. Haag. *Local quantum physics*. Springer, Berlin, 1993.

[19] J. A. Holbrook, D. W. Kribs, and R. Laflamme. Noiseless subsystems and the structure of the commutant in quantum error correction. *Quantum Information Processing*, 2:381–419, 2003, arXiv:quant-ph/0402056.

[20] E. Knill and R. Laflamme. Theory of quantum error-correcting codes. *Phys. Rev. A*, 55:900–911, 1997.

[21] E. Knill, R. Laflamme, and L. Viola. Theory of quantum error correction for general noise. *Phys. Rev. Lett.*, 84:2525–2528, 2000.

[22] K. Kraus. *States, Effects and Operations: Fundamental Notions of Quantum Theory*. Springer-Verlag, Berlin, 1983.

[23] D. Kribs, R. Laflamme, and D. Poulin. Unified and generalized approach to quantum error correction. *Physical Review Letters*, 94(18):180501, 2005.

[24] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky. Operator quantum error correction. *Quantum Information and Computation*, 6:382–399, 2006, arXiv:quant-ph/0504189.

[25] D. W. Kribs and R. W. Spekkens. Quantum error-correcting subsystems are unitarily recoverable subsystems. *Physical Review A*, 74(4):042329, 2006, arXiv:quant-ph/0608045.

[26] S. Lloyd and J.-J. E. Slotine. Analog quantum error correction. *Phys. Rev. Lett.*, 80(18):4088–4091, 1998.

[27] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, The Pitt Building, Trumpington Street, Cambridge, UK, 2000.

[28] V. I. Paulsen. *Completely Bounded Maps and Operator Algebras*. Cambridge University Press, 2002.

[29] D. Poulin. Stabilizer formalism for operator quantum error correction. *Phys. Rev. Lett.*, 95:230504, 2005.

[30] P. W. Shor. Scheme for reducing decoherence in quantum computer memory. *Phys. Rev. A*, 52(4):R2493–R2496, 1995.

[31] A. M. Steane. Error correcting codes in quantum theory. *Phys. Rev. Lett.*, 77(5):793–797, 1996.

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